Thesis/
Reports
Brewer, J.W.

OPTIMAL IMPULSIVE SILVICULTURE IN THE CHAPMAN-RICHARDS MODEL FOREST

by
John W. Brewer

Optimal Impulsive Silviculture In The Chapman-Richards Model Forest

A Final Report To The Forest Service,
United States Department of Agriculture

By:
John W. Brewer
Professor of Mechanical and Aeronautical Engineering,
Bainer Hall,
University of California
Davis, CA 95616

Cooperative Agreement No. 28-C2-657 With
Rocky Mountain Forest And Range Experiment Station
Contracting Section
240 West Prospect Street
Fort Collins, CO 80526-2098
21 February, 1995

September 1994

LIBRARY COPY
ROCKY MTN. FOREST & RANGE
EXPERIMENT STATION

Introduction

A fundamental approach in the modelling of economic behavior is the use of optimal control theory (Intriligator, 1971). The applied person assumes that an understanding of behavior, gained in this way, might be a basis for the formulation of management policy. In particular, the theoretical problems of forest management have been approached in this way. In some studies, the tools are quite complicated including simulation models that describe individual trees and all that under the supervision of large nu-merical optimization programs. That is fine for numerical results but the procedure becomes intract-able where the goals are explicit and analytic principles of economic behavior and management policy.

In this paper, we follow an alternative approach, used by some others, in which a more tractable model is used to obtain these principles (Betters et al., 1991; Donnelly and Betters, 1991; Reed and Apaloo, 1991). These studies are the application of the maximum principle to a continuous-time model of forest growth and their results are quite complete in that context. Our primary contribution is to restate this study in a discrete-time context and model the tree-cutting operations as "impulses"; that is, as operations of extremely short duration compared to the time intervals between decision periods and short compared to the times scales of biological growth. In this way, the true nature of forest operations is more accurately modeled and the effects, say, of the length of time between inventories and management decisions can be investigated. This discrete decision model is in the realm of dynamic programming; however, our approach is analytical and we mention the use of programming only to confirm our theoretical results.

Also, we are able to provide some interesting examples of the variety of economic behavior that can be expected in the forest setting as well as the effect of particular management policies.

The overall approach taken here is much like that used in previous studies (Brewer, 1981, 1993).

The model forest

Forest scientists have described a forest-growth model of some generality, the *Generalized Chapman-Richards Equation* (for example, see Betters et al., 1991; Donnelly and Betters, 1991; Donnelly et al ,1992):

$$\frac{d\xi}{dt} = \frac{rt^{-b}}{m-1}\xi\left[1-\left(\frac{\xi}{X_{max}}\right)^{m-1}\right]-\eta(t) \qquad (1)$$

Here, "ξ" represents the commercial wood volume in cubic measure per unit of land area and "η" represents the harvest rate. The parameters r, m, b, and X_{max} are empirical constants that are determined for a particular species of tree as well as the micro-climate and bio-chemical circumstances of the location (the "site").

This is a single differential equation with some unique and interesting features. For example, density-dependent competition is accounted for with a logistic-like term. The parameter "X_{max}" is the maximum value of "ξ" that can be supported on the given area of land. The parameter "r" is a proportionality constant and is interpreted as an inherent growth rate. The term, t^{-b}, is a growth attenuation factor, we say "attenuation" because we expect that $0 \le b \le 1$. One includes this term in order to model a natural tree senescence. Many well-known growth equations correspond to special values of the parameter "m", the *von Bertalanffy model*, a *time-dependent Gompertz equation*, and a *time-dependent logistic equation* to mention just a few. We expect $m > 0$. The case where $m = 1$ is a limit case that must be handled with special mathematical procedures.

It is easy to verify the analytic solution of the Chapman-Richards equation (1) where $\eta \equiv 0$ (we say *free growth*):

$$\frac{\xi(t)}{X_{max}} = \left(1+\left[\left(\frac{\xi(t_0)}{X_{max}}\right)^{1-m}-1\right]\exp\left[-\frac{r}{1-b}\left(t^{1-b}-t_0^{1-b}\right)\right]\right)^{\frac{1}{1-m}} \qquad (2)$$

Reformulation in discrete-time

Begin by dividing time into discrete intervals of equal length Δt. It is assumed that at the end of each interval, an inventory of the stand is made and that is followed by a decision for the management of the stand in the next interval. That is to say, decisions are made at the beginning of each of these intervals. Adopt the notations

$$t_k = t_0 + k \cdot \Delta t \tag{3}$$

and

$$\xi_k = \xi(t_0 + k \cdot \Delta t), \qquad 0 \le k \le K \tag{4}$$

where "k" is an integer. Assume that the decision is made at each interval to either

(1) thin the stand,

or (2) clear-cut,

or (3) allow further free growth.

Also, assume that the duration of a thinning or clear-cut operation is extremely small on any relevant biological time scale. Then, a reasonable description of the forest operation is

$$\eta(t) = \eta_0 \, \delta(t-t_0) + \eta_1 \, \delta(t-t_1) + \eta_2 \, \delta(t-t_2) + \ldots + \eta_K \, \delta(t-t_K) \tag{5}$$

where $\delta(t-t_k)$ represents a Dirac delta or "impulse" at $t = t_k$. The result of the operation is

$$\xi_k^{+} = \xi_k^{-} - \eta_k \tag{6}$$

where the "–" superscript means the value prior to an impulsive harvest at time t_k and the "+" superscript indicates the value immediately following the operation at the same instant of time.

It follows that solution (2) can be rewritten as

$$x_{k+1}^{-} = \left(1 + \left[\left(x_k^{-} - h_k\right)^{1-m} - 1\right]\Delta_k\right)^{\frac{1}{1-m}} \tag{7}$$

where

$$\Delta_k = \exp\left[-\frac{r}{1-b}\left(t_{k+1}^{1-b} - t_k^{1-b}\right)\right] \tag{8}$$

and where we have introduced the normalized variables

$$x_k = \xi_k / X_{max} \qquad h_k = \eta_k / X_{max} \tag{9}$$

More generally

$$x_k^{-} = \left(1 + \left\{\left(x_j^{-} - h_j\right)^{1-m} - 1\right\}A(j,k)\right)^{\frac{1}{1-m}} \tag{10}$$

where

$$\Delta(j,k) = \exp\left\{\frac{r}{1-b}\left[t_k^{1-b} - t_j^{1-b}\right]\right\} \tag{11}$$

Notice that

$$(t_k)^{1-b} - (t_j)^{1-b} = (t_k)^{1-b} - (t_{k-1})^{1-b} + (t_{k-1})^{1-b} - (t_{k-2})^{1-b} \cdots - (t_{j+1})^{1-b} + (t_{j+1})^{1-b} - (t_j)^{1-b}$$

so

$$\Delta(j,k) = \prod_{\ell=j}^{k-1} \Delta_\ell \tag{12}$$

The model of economic behavior

The model of economic behavior in the forest is that of an optimal silviculture - The maximization of the current value of forest asset. The "current value" is composed of the profits less the costs of the stand, accumulated and discounted over all time. Profits are obtained from sales during thinning of the stand and, of course, from a final clear-cut.

Fundamental to our approach is the notion that the land is to be used in perpetuity for forest cultivation. The idea is that after a clear-cut at $k = K$, the forest is regenerated and the entire process begun again. (An example of a "cost" is the cost of cultivating the land and replanting the stand during regeneration.) It is interesting that we do not necessarily condemn the actors to engage in this "infinite-series" type of silviculture if we allow the final decision time, K, to be a free parameter. That is to say, if it turns out that the optimal value of $K \to \infty$, the actual optimal silviculture will be a single period of cultivation. In fact, that is the result for some sets of parameter values.

The optimization criterion

Denote the present value of the forest asset by "V" and recognize that

$$\frac{V}{X_{max}} = E_K\, e^{-\delta(t_0+K\Delta t)}\left(q_K\, x_K - \frac{C}{X_{max}}\right) + E_K \sum_{k=0}^{K-1}\left(e^{-\delta(t_0+k\Delta t)} p_k\, h_k\right) \tag{13}$$

where "δ" is the discount rate and

$$E_K = 1 + e^{-K\Delta t} + e^{-2K\Delta t} + e^{-3K\Delta t} + \cdots = \frac{1}{1-e^{-K\Delta t}} \tag{14}$$

This is the standard form of the factor used to represent the fact that the land is to be after each occurrence of $k = K$ and the whole process begun again. "C" represents the regeneration cost. The number "q_k" represents the unit price less unit cost associated with the sale of the wood after a clear-cut and "p_k" represents the same thing after a thinning operation. Market experience is that these two numbers are not the same.

The subscript "k" on the "p" and "q" is an acknowledgment that these prices change with changing market conditions and with technological advance. We quantify the effects of the model market and technology changes with

$$q_k = Q_0 \exp(\rho[t_0 + k\cdot\Delta t]) \qquad \text{and} \qquad p_k = P_0 \exp(\rho[t_0 + k\cdot\Delta t]) \qquad (15)$$

where "ρ" is taken to be some constant. The most immediate reason is that as the forest stand grows, the tendency is toward the more valuable, larger-diameter trees. We can have some confidence in assuming that the sales price of a harvested stand will continue to increase in somewhat the same way even after the tree diameters have obtained that constant value consistent with senescence and maximum density, X_{max}. The justification is that even with a slow-down in world population growth, the expectations and the means of individuals will continue to rise for some time to come. Also, we can expect continued reduction in costs due to technological advance. The complete modeling of these factors will always be the most challenging (and most controversial) aspect of optimal silviculture. We take the simplest imaginable approach here.

The discount rate, δ, is used to evaluate the asset against alternative uses of capital. It turns out that it can also be used in certain instances to account for fire hazard although that is not the case where the thinning operations themselves affect the hazard rate (Reed and Apaloo, 1991). One expects that the same sorts of thing can be said of the hazard of infestation.

Finally, there is the constraint

$$0 \le h_k \le x_k \qquad 0 \le x_k \qquad (16)$$

The Lagrange function and necessary conditions

First, divide equation (13) by Q_0. Then, recognize that the impulsive Chapman-Richards equation (7) is a constraint that must be satisfied at each "k" and form the Lagrange function:

$$F = F_K + \sum_{k=0}^{K-1}\left[E_K\ \mathrm{Re}^{(\rho-\delta)t_0}\ \phi^k h_k + \psi_k\left(a_k - x_{k+1}\right)\right] \qquad (17)$$

where

$$\phi = \exp([\rho-\delta]\Delta t), \qquad R = P_0/Q_0, \qquad (18)$$

$$F_K = E_K\ [\exp\{(\rho-\delta)t_0\}\phi^K x_K - \exp(-\delta(t_0+K\Delta t)S], \qquad S = C/(X_{max}\,Q_0) \qquad (19)$$

We refer to "F_K" as a *terminal benefit*. At this point, the criterion is fully normalized, "F" and "F_K" are dimensionless.

It is convenient to place x_K outside the summation sign; so, make the adjustment

$$F = F_K + \psi_K x_K - \psi_0 x_0 + \sum_{k=0}^{K-1}\left[E_K\, Re^{(\rho-\delta)t_0}\,\phi^k h_k + \psi_k a_k - \psi_k x_k\right] \tag{20}$$

The necessary conditions for optimality are well known: There is

$$\partial F/\partial h_k = E_K R\phi^k \exp(\{\rho-\delta\}t_0) + \psi_{k+1}\,\partial a_k/\partial h_k = 0 \tag{21}$$

where

$$\frac{\partial a_k}{\partial x_k} = \frac{-(x_k - h_k)^{-m}\Delta_k}{\left(1 + \left\{(x_k - h_k)^{1-m} - 1\right\}\Delta_k\right)^{\frac{m}{n-1}}} \tag{22}$$

So, the thinning control, h_k, must be such that

$$\frac{\psi_{k+1}(x_k - h_k)^{-m}\Delta_k}{\left(1 + \left\{(x_k - h_k)^{1-m} - 1\right\}\Delta_k\right)^{\frac{m}{n-1}}} = \psi_{k+1}\Delta_k\left(\frac{x_{k+1}}{x_k - h_k}\right)^m = E_K R\phi^k e^{(\rho-\delta)t_0} \tag{23}$$

Where h_k can meet this condition without a violation of the constraints, a thinning operation in the form of an impulsive harvest is the proper decision.

In any case, it is necessary that

$$\partial F/\partial x_k = \psi_{k+1}\,\partial a_k/\partial x_k - \psi_k = 0$$

so

$$\frac{\partial F}{\partial x_k} = \frac{\psi_{k+1}(x_k - h_k)^{-m}\Delta_k}{\left(1 + \left\{(x_k - h_k)^{1-m} - 1\right\}\Delta_k\right)^{\frac{m}{n-1}}} - \psi_k = 0 \tag{24}$$

This is a difference equation that must be satisfied by the Lagrange multipliers at all values of "k".

The evolution of the multipliers during thinning

Combine the necessary conditions, (23) and (24), to conclude that on the thinning trajectory

$$\psi_k = \psi_{k+1}\Delta_k\left(\frac{x_{k+1}}{x_k - h_k}\right)^m$$

$$= \psi_{k+1}\Delta_k\left(\frac{1}{(\phi\Delta_k)^{\frac{1}{m}}}\right)^m$$

That is,

$$\psi_{k+1} = \phi\psi_k \tag{25}$$

Also,

$$\psi_k = E_K R\phi^k \exp[(\rho-\delta)t_0] \tag{26}$$

The evolution of multipliers during free growth

On the other hand, we have the multiplier equation (24) during free growth:

$$\psi_{k+1}\Delta_k\left(\frac{x_{k+1}}{x_k}\right)^m = \psi_k \qquad (27)$$

For example, suppose that a switch to free growth occurs at sample j and terminates at sample K; then, the solution to the multiplier equation is

$$\psi_j = \Delta_j\left(\frac{x_{j+1}}{x_j}\right)^m \Delta_{j+1}\left(\frac{x_{j+2}}{x_{j+1}}\right)^m \cdots \Delta_{K-1}\left(\frac{x_K}{x_{K-1}}\right)^m \psi_K$$

$$= \left(\frac{x_K}{x_j}\right)^m \Delta(j,K)\psi_K \qquad (28)$$

The thinning trajectory

Combine (24) and (27):

$$\frac{\phi^{k+1}(x_k - h_k)^{-m}\Delta_k}{\left(1 + \left\{(x_k - h_k)^{1-m} - 1\right\}\Delta_k\right)^{\frac{m}{m-1}}} = \phi^{k+1}\Delta_k\left(\frac{x_{k+1}}{x_k - h_k}\right)^m = \phi^k \qquad (29)$$

This result is used to show two things: (1) The stand value before harvest on the thinning trajectory

$$x_{k+1}^- = D_k^{1/(1-m)}\left(x_k^- - h_k\right) \qquad \text{where } D_k = \left[\phi\Delta_k\right]^{(m-1)/m} \qquad (30)$$

and, (2) The stand value after harvest

$$x_k^+ = x_k^- - h_k = \left(\frac{1-\Delta_k}{D_k - \Delta_k}\right)^{\frac{1}{1-m}} \qquad (31)$$

Substitute into (31) into (30) and find that

$$x_{k+1}^- = \left(D_k\frac{1-\Delta_k}{D_k - \Delta_k}\right)^{\frac{1}{1-m}} \qquad (32)$$

Also, the optimal control during thinning is given by

$$h_k = x_k^- - x_k^+ = \left(D_{k-1}\frac{1-\Delta_{k-1}}{D_{k-1} - \Delta_{k-1}}\right)^{\frac{1}{1-m}} - \left(\frac{1-\Delta_k}{D_k - \Delta_k}\right)^{\frac{1}{1-m}} \qquad (33)$$

Blocking conditions

There are several interesting situations where the optimal control is $h_k < 0$. A violation of a basic constraint. The convention is to refer to this as a *blocking condition* and the corresponding interval of time as a *blocked interval.*

A block often occurs if $\rho > \delta$ and the explanation is quite simple and interesting. The growing demand and prices are outstripping the discounting so that the forest stand is becoming an attractive investment. Becoming such an attractive investment that the optimal control is to transplant trees of the current age and thereby take further advantage of these favorable rates. This transplanting is the meaning of negative harvest. Of course, the costs of transplanting are prohibitive and not accounted for in our formulation. Rather than add such an accounting, we merely adjust the harvest rate to zero in all blocked intervals. This assumption is justified in our treatment of the Kuhn-Tucker conditions.

Terminal conditions

There are some additional necessary conditions that can be exploited in an effort to find that period, $k = K$, when the land is clear cut and the replanting is begun. It must be true that

$$\partial F/\partial x_K = \exp\{(\rho-\delta)t_0\}E_K \phi^K - \psi_K = 0 \tag{34}$$

So, we have a condition on the terminal value of the multiplier:

$$\psi_K = \exp\{(\rho-\delta)t_0\}E_K \phi^K \tag{35}$$

It is also true that we must require that

$$\partial F/\partial \psi_K = -x_K + a_{K-1} = 0 \tag{36}$$

However, this adds no new information since it is merely a restatement of the state equation.

Clear-cut condition during thinning

Notice it is possible that the optimal control calls for $x_k^+ = x_k^- - h_k < 0$. We interpret this to mean that the stand is thinned out of existence. The interesting question is does this mean that $k = K$, that the clearcut and replanting is to begin? The answer is this is *not* always the case.

Let us examine this situation a bit further with a study of (31). If $m < 1$, $x_k^+ = x_k^- - h_k < 0$ would mean that $\Delta_k = 1$, which is impossible. Thus, a clear-cut can occur during thinning only if $m > 1$ and the condition for that is

$$\Delta_k = \phi^{m-1} \tag{37}$$

This condition will obtain if and only if $\rho < \delta$. Where $t_0 = 0$, this condition reduces to

$$(k+1)^{1-b} - k^{1-b} = \frac{(\delta-\rho)(m-1)(1-b)}{r}(\Delta t)^b \tag{38}$$

We have derived two terminal conditions must be satisfied simultaneously in order for $k = K$: There is the terminal multiplier condition certain (35) and one other condition that is obtained in the sequel (see (??)). Before that derivation begins, it is useful to make an observation: Often these condition will not be met simultaneously so that $K > k$, the value obtained from (38). However, that

observation raises the next question: How can the present value increase after the stand has been clear-cut since there is no more wood for sale? The answer is obtained by a consideration of the other side of the picture, the costs. It is possible to increase the present value, V, by reducing cost and that is done in some instances of optimal silviculture by delaying - *hence, discounting* - the replanting costs characterized by "C". The point is that discounting replanting costs by moving "K" into the future can lead to a non-decreasing rate in present value of the stand *ad infinitum*, $K \rightarrow \infty$. An interpretation is that the theory of optimization overrules us and automatically converts the problem solution into a single period prescription!

Two conditions for a terminal impulsive harvest (clear-cut)

One condition for $k = K$ is the terminal multiplier condition (35). In words, this is the condition that the terminal value of "x" is a stationary value for the present value. We can obtain a second terminal condition by considering variations in the upper limit of the summation sign in the definition of the present value (13). This second condition must be satisfied simultaneously with (35) in order that a particular "k" be a candidate for the terminal sample "K".

The increment in "F" corresponding to an increase of K to $K+1$ is denoted Δ_F. Since the final time occurs after at least one period of free growth, we can take h_K equal to zero so that

$$\Delta_F = E_{K+1}[\exp\{(\rho-\delta)t_0\}\phi^{K+1}x_{K+1} - \exp\{-\delta(t_0+(K+1)\Delta t)\}S]$$
$$-E_K[\exp\{(\rho-\delta)t_0\}\phi^K x_K - \exp\{-\delta(t_0+K\Delta t)\}S] \leq 0 \qquad (39)$$

or

$$S(E_K - e^{-\delta\Delta t}E_{K+1}) \leq \exp[\rho(t_0+K\Delta t)][E_K x_K - E_{K+1}\phi x_{K+1}] \qquad (40)$$

The economic interpretation is interesting: The left side of (40) is the economic gain from delaying, *hence discounting*, the replanting costs and the right side is the loss in added-value that results from not growing an additional period.

We can expand upon the terminal conditions. Suppose that a switch to free growth occurs at sample j and there is a clear-cut at sample K; then, (28) becomes

$$\psi_j = \left(\frac{x_K}{x_j}\right)^m \Delta(j,K)\psi_K$$

So, where the terminal condition on the multiplier obtains,

$$R\phi^j = \left(\frac{x_K}{x_j}\right)^m \Delta(j,K)\phi^K \qquad (41)$$

Thus, the locus of points for the terminal condition on the multipliers

$$x_K = x_j\left(\frac{R}{\phi^{K-j}\Delta(j,K)}\right)^{1/m} \tag{42}$$

We have the solution of the dynamic equation

$$x_K/x_j = \{ x_j^{m-1} + (1 - x_j^{m-1})\Delta(j,K)\}^{1/(1-m)} \tag{43}$$

and the after-thinning value (31)

$$x_j^+ = \left(\frac{1-\Delta_j}{D_j - \Delta_j}\right)^{\frac{1}{1-m}}$$

So

$$x_K = \left[\frac{D_j - \Delta_j + (1-D_j)\Delta(j,K)}{D_j - \Delta_j}\right]^{\frac{1}{1-m}}$$

and so

$$\frac{x_K}{x_j} = \left[\frac{D_j - \Delta_j + (1-D_j)\Delta(j,K)}{1-\Delta_j}\right]^{\frac{1}{1-m}} \tag{44}$$

Terminal multiplier condition (41) becomes

$$D_j - \Delta_j + (1-D_j)\Delta(j,K) = (1-\Delta_j)\left(\frac{\phi^{K-j}\Delta(j,K)}{R}\right)^{\frac{m-1}{m}} \tag{45}$$

Optimal impulsive silviculture at the initial time and the Kuhn-Tucker conditions

There is a final remaining question: What is the optimal procedure at t_0? This question must be resolved for two possibilities: $x(t_0) > (x_0)_{thining}$ and $x(t_0) < (x_0)_{thining}$. In the first case, we merely apply the necessary condition (31):

$$h_0 = x(t_0) - \left(\frac{1-\Delta_0}{D_0 - \Delta_0}\right)^{\frac{1}{1-m}}, \qquad x(t_0) > x_0^- \tag{46}$$

That is to say, the optimal initial harvest is an impulsive treatment that thins the stand to the level of the thinning trajectory.

The final case is where $x(t_0) < (x_0)_{thining}$, the situation, say, where the stand has been aquired in a recently regenerated state. It will be demonstarted that the optimal procedure is allow free

growth of the stand up to the level of the thinning trajectory. To address this interesting case, we turn to the classical Kuhn-Tucker conditions. Introduce the slack variables

$$s_k^{\,2} = h_k \tag{47}$$

and in the classical fashion treat this as an equality constraint. This means that the modified Lagrange function is

$$F = F_K + \sum_{k=0}^{K-1}\left[E_K R e^{(\rho-\delta)t_0}\phi^k h_k + \psi_{k+1}(a_k - x_{k+1}) + \mu_k\left(s_k^2 - h_k\right)\right] \tag{48}$$

Where the "μ_k" are additional multipliers. There is an additional necessary condition

$$\partial F/\partial s_k = 2 s_k \mu_k = 0 \tag{50}$$

This is satisfied by one of only two possibilities: Either (1) $s_k = 0$, so the free-growth constraint is satisfied ($h_k = 0$), or (2) the constraint is not satisfied and $\mu_k = 0$. In the latter case, thinning occurs and the development poceeds exactly as before. In the first case, necessary condition (50) becomes

$$\mu_k = \psi_{k+1}\Delta_k\left(\frac{x_{k+1}}{x_k}\right)^m - E_K R\phi^k e^{(\rho-\delta)t_0}$$

We can rewrite this condition as

$$\mu_k = (\psi_k)_{\text{free}} - (\psi_k)_{\text{thinning}} \tag{51}$$

There is the classic result that, in the case of a feasible change at the constraint boundary, $\Delta h_k \geq 0$, the result is the change in the optimal value of the criterion, $\text{sign}(\Delta V_{max}) = \text{sign}(\mu_k)$. Thus, the present value is maximized at the constraint boundary (free growth is optimal) only if

$$\mu_k \leq 0 \tag{52}$$

Combine (51) and (52) to conclude that the multilpier, ψ_k, acts as a switching function during the initial phase of optimal impulsive silviculture. Next, observe the backward difference equation (27):

$$\psi_k = \psi_{k+1}\Delta_k\left(\frac{x_{k+1}}{x_k}\right)^m$$

We argue that this means that the optimal initial silviculture is free growth where $x(t_0) < (x_0)_{thining}$. The reason is that the coefficient of ψ_{k+1} is < 1 so, if k+1 is the sampling time where the free-growth trajectory crosses the thinning trajectory, then $\psi_k < \psi_{k+1}$. The optimality of free growth then follows from the Kuhn-Tucker condition (52). This result is obtained in an iterative way for the x_k-trajectory all the way back to the initial condition.

Examples

The use of the conditions for optimal impulsive control of a forest stand are illustrated in this section.

Example 1: Donnelly and Betters (1991) studied the situation where the biological parameters

$$m = 1.11, \quad r = 0.155, \quad b = 0.5, \quad K = 892 \text{ m}^3/\text{ha}$$

and the economic parameters

Q_0 = US$ 0.883 per cubic meter, P_0 = US$ 0.75 per cubic meter,

$\delta = 0.04$ (4% per annum), $\rho = 0.02$,

$S = 0$

In addition, we take

$$\xi(t_0) = 0.13 \text{ m}^3/\text{ha}$$

All this is suggestive of a even-aged stand of lodgepole pine (*Pinus Contorta Dougl.*). The crucial sub-arcs of this optimal prescription are typical, a free growth followed by a thinning sub-trajectory and then by a final switch to free growth concluding in a clear-cut at a time that satisfies both terminal conditions (35) and (40). The results for various values of Δt are provided in Table 1.

Table 1: The Results Of Example 1

For Various Values of Δt

Time Between Inventories (Years)	Switch To Thinning-Type Impulses In Year	Switch To Free-Growth In Year ...	Clear-Cut And Regenerate In Year ...
1/2	40.5	59	81.5
1	40	59	81
5	40	60	80
10	40	80	100
20	40	120	140

The graphs of two of these optimal-impulsive trajectories are presented in Figures 1 and 2. The results indicate a clear dependency of the results on the time between inventories although the dependency is not dramatic until we reach the value of $\Delta t = 10$ years.

The results were confirmed with the use of dynamic programming. A second successful test was a comparison of the well-known continuous-time results with our results for optimal impulsive control at low values of Δt.

Example 2: In this second example, we obtain results where

$P_0 = Q_0$, $b = 0.4$

$\delta = 0.02$, $\rho = 0.045$,

C/K = US$ 0 per cubic meter, $x(t_0) = 0.25$

All other parameters are selected as in Example 1. This is a subcase where a blocking interval can occur. The curves that characterize the situation are free growth, the thinning trajectory, and the terminal trajectory. These are presented in Figure 3. There is a switch to thinning in year 10 and a switch back to free growth in year 50. This second switch is necessitated by the fact that an unconstrained optimal value of h_k would be less than zero in this year. This is no terminal harvest and the present value of the stand continues to grow indefinitely much as the terminal clear-cut is imagined to proceed into the future. This is much as a high-interest bank account that is never withdrawn. In a sense then, the infinite-series solution is circumvented since K approaches infinity.

Conclusions

Formula (7) is used to obtain an exact reformulation of the problem of optimal silviculture into a problem of optimal impulsive control in discrete time. We used standard calculus and the necessary conditions of Kuhn and Tucker to solve this optimal silviculture problem and the results are quite complete. The examples indicate some similarities between optimal-impulsive control and the solution to the continuous-time silviculture problem obtained by others (Betters et al (1991); Donnelly et al

Figure 1

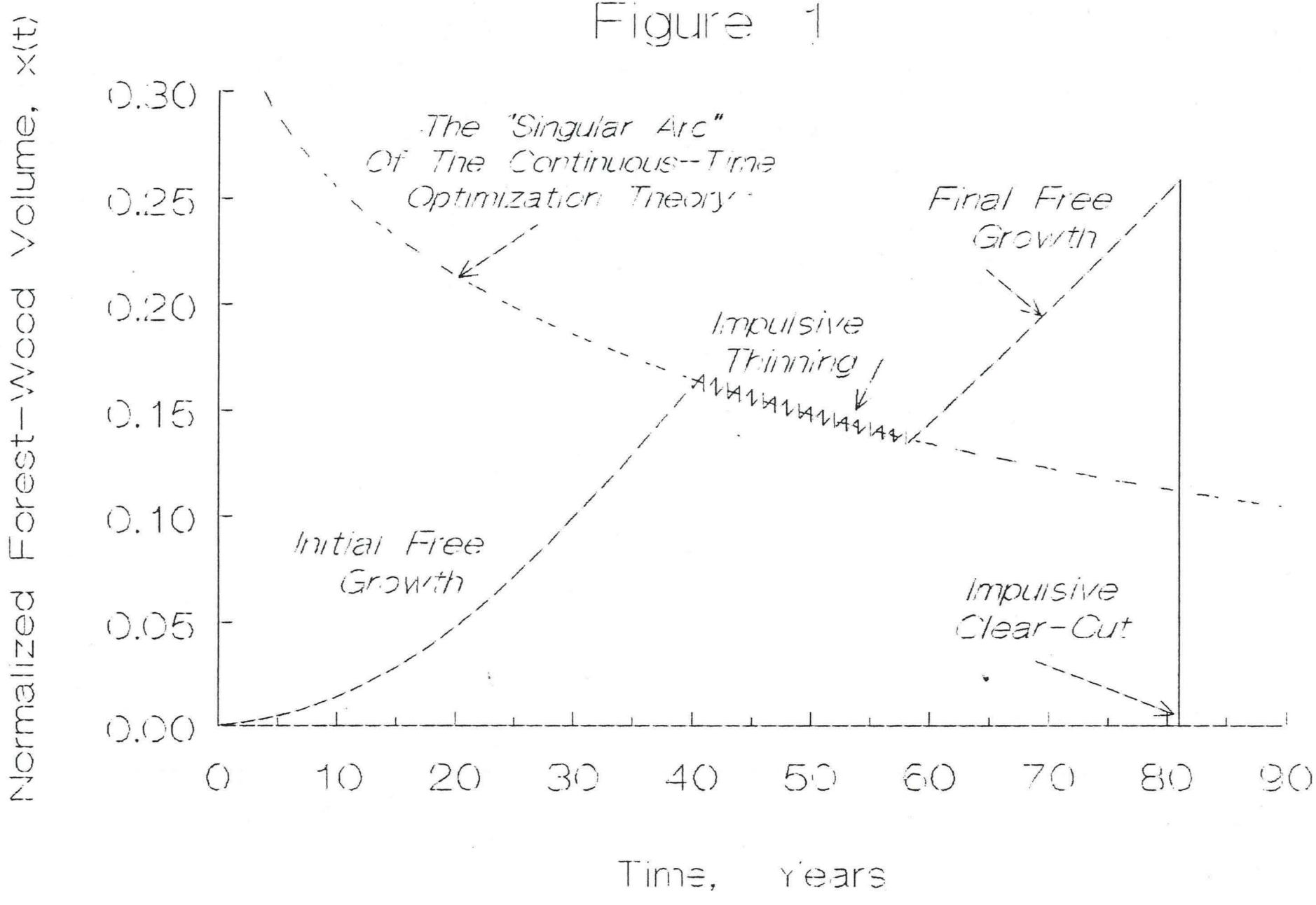

Normalized Forest-Wood Volume, x(t)
0.30
0.25
0.20
0.15
0.10
0.05
0.00
0
10
20
30
40
50
60
70
80
90
Time, Years
The "Singular Arc"
Of The Continuous-Time
Optimization Theory
Final Free
Growth
Impulsive
Thinning
Initial Free
Growth
Impulsive
Clear-Cut

Figure 2

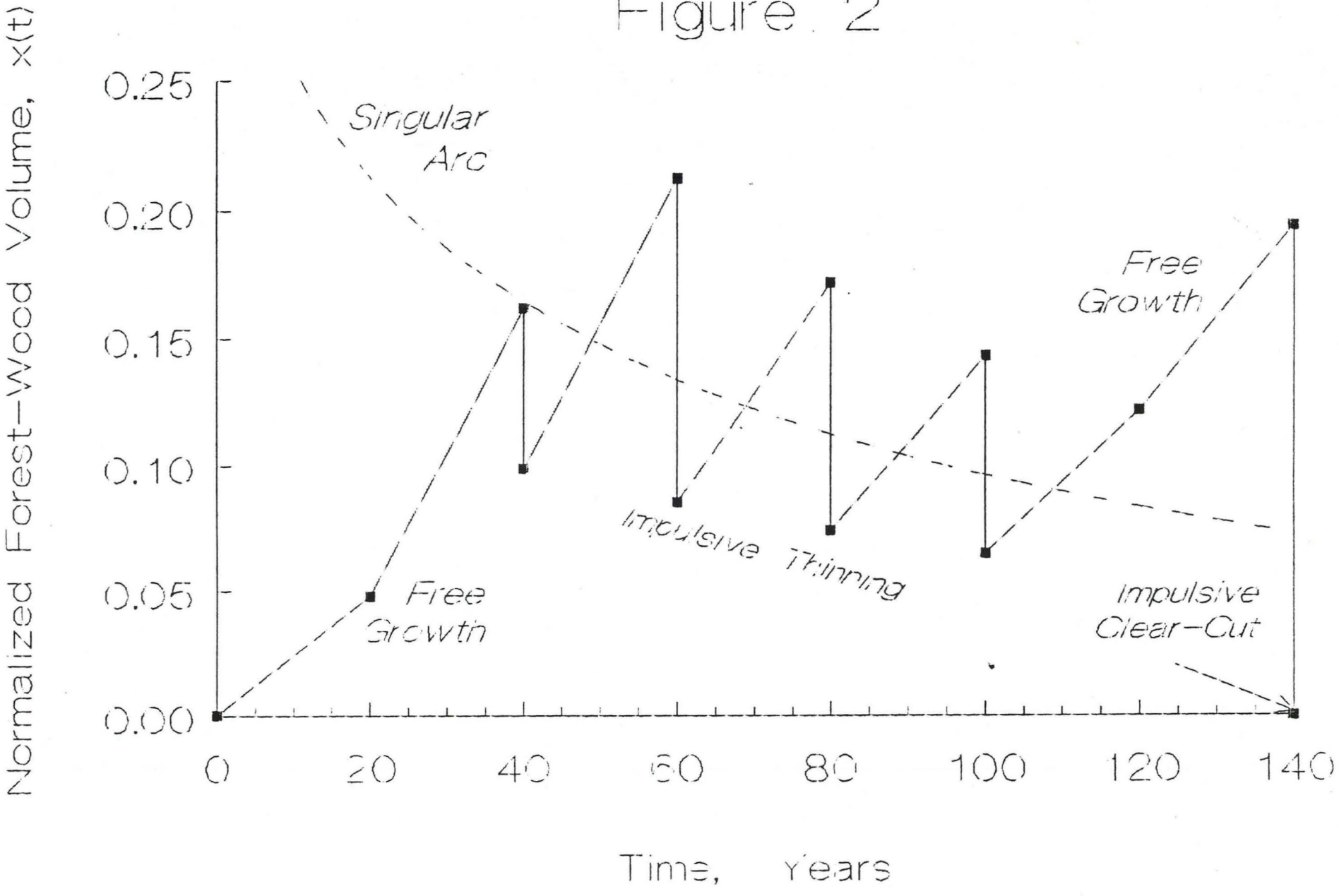
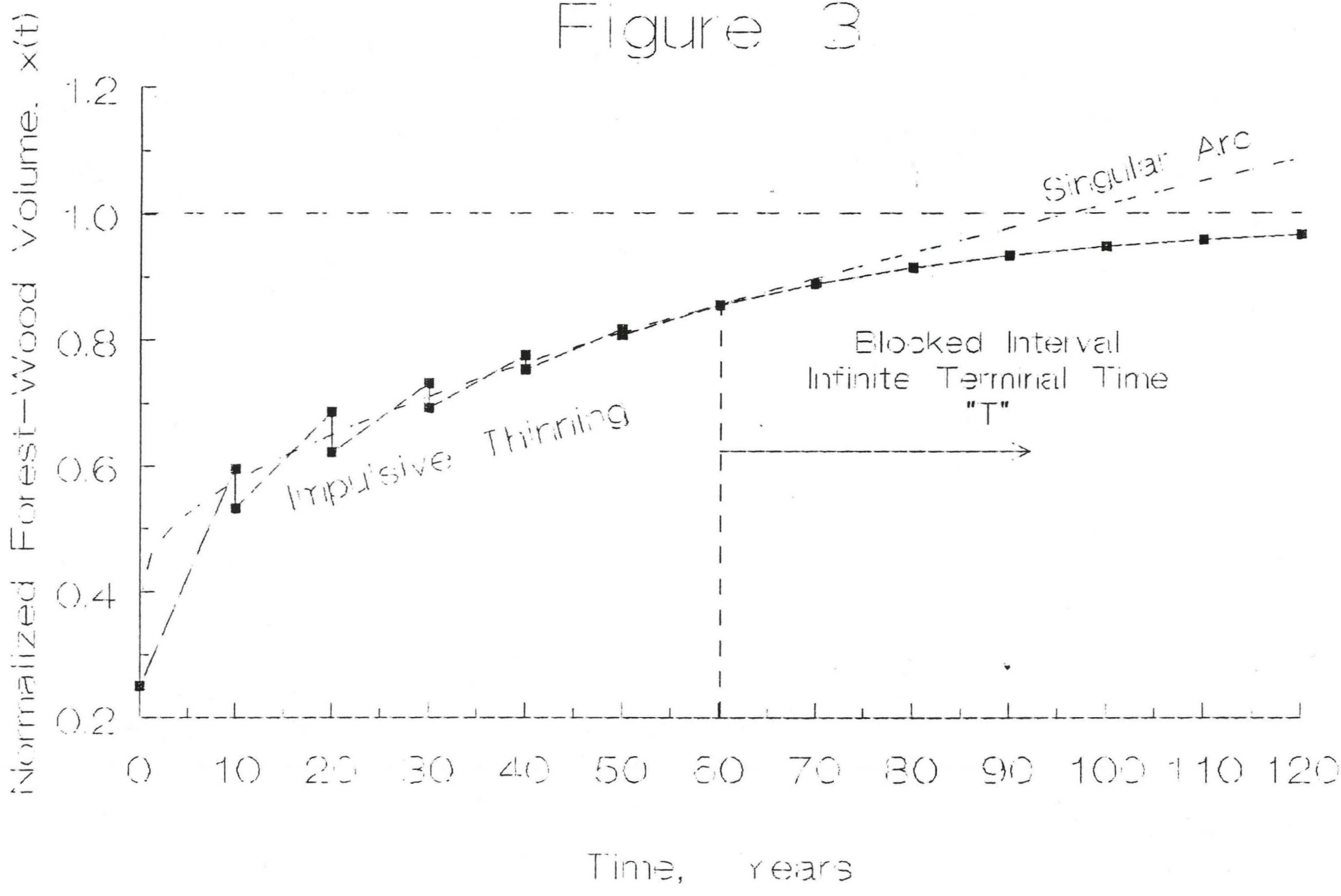

Normalized Forest-Wood Volume, x(t)
0.25
0.20
0.15
0.10
0.05
0.00
0
20
40
60
80
100
120
140
Time, Years
Singular Arc
Free Growth
Impulsive Thinning
Free Growth
Impulsive Clear-Cut

Figure 3


Normalized Forest-Wood Volume, x(t)
1.2
1.0
0.8
0.6
0.4
0.2
0
10
20
30
40
50
60
70
80
90
100
110
120
Time, years
Singular Arc
Blocked Interval
Infinite Terminal Time
"T"
Impulsive Thinning

(1992)). There are some interesting and important dissimilarities; in particular, there is the effect of the time between inventories, Δt, on the optimal timing of operations such as a clear-cut.

We displayed several instances where a clear-cut is contra-indicated and that is a surprising and important observation.. Where $\rho > \delta$, the present value of the forest represents its sales value and a clear-cut can be contra-indicated much as the closing of any productive investment might be. This can be true even in a climax forest, $\xi = X_{max}$. Technically, a clear-cut can be contra-indicated even where $\rho < \delta$, but in that case the forest will be completely harvested by a continual thinning rather than by a final large impulse, by what we call a "clear-cut".

There is no comprehensive attempt in our research to account for the effects of a variety of fixed costs. However, there is one fixed cost that has an important effect on our version of optimal silviculture - the cost of regeneration "C". For example, one way to increase the "present value" is to reduce cost and that is done in some instances of optimal silviculture by delaying - *hence, discounting* - replanting costs. The point is that discounting replanting costs by moving "K" into the future can lead to a nondecreasing rate *ad infinitum* in present value of the stand. Sadly, this may in fact describe behavior in the unregulated forest. This happens only under special circumstances and when it happens the theory of optimization tells us that $K \rightarrow \infty$. The interpretation is that the theory of optimal silviculture overrids our original formulation and automatically converts the problem solution into a single period prescription.

Even in our simplified model, site quality seems to effect the onset of thinning and the rotation age "K", but through the parameters "b" and "r" rather than "X_{max}". That is, it is the vigor of the stand, and not the maximum supportable density, that is important in this way. That conclusion is based upon the fact the timing of events can be completely obtained in terms of the non-dimensionlized variables, x_k and h_k.

Optimal-impulsive feedback control

Rewrite Eqn(31) as

$$-h_k = r_k - x_k \tag{53}$$

Re-interpret "x_k" to be the newly measured value of the forest stand, $\xi(t_k)/X_{max}$, obtained from the a recent inventory at t_k. The first term is thought of as a "reference signal" or "setpoint":

$$r_k = \left(\frac{1 - \Delta_k}{D_k - \Delta_k} \right)^{\frac{1}{1-m}} \tag{54}$$

That is, we interpret (31) as a *feedback control.* It is surprising to discover that the optimal impulsive control of the nonlinear, time-varying model forest reduces to a linear feedback rule.

This interpretation of our results holds out the promise of a robust control if, in fact, the silviculture is actually implimented as a feedback control.

Perspective

There are those that see optimal silviculture not as a design method; rather, this optimization tool is seen as a way to understand market behavior. The point being that inefficient producers are forced out of the market so that competition leads to a maximization of the present value of all capital assets, and that leads to the type of behavior described by the optimal silviculture problem. To the extent that the latter interpretation is valid, the results presented in this paper are of more use to the regulation function than to management. The regulator can impose fees or rents which affect the values of "p" and "q" or the regulator can delineate the nature of the regeneration and that will affect the value of "C". In this way, the regulator can use natural market mechanisms to control economic behavior in the forest. That interpretation is probably the best point-of-view at the current state of the art.

The discount rate, δ, is never known with any certainty and it is likely that much will be learned with the study of a stochastic optimization wherein this rate is known only to within a probability distribution. There are other reasons to believe that a stochastic optimization will be a important

addition to our understanding of management prescriptions and market behavior. There is some probability that the stand will be destroyed at some point by fire, infestation or disease. The evolution of these probabilities will influence the present value of the stand and will have an effect on the optimal prescription. Reed and Apaloo (1991) provide preliminary understanding of these matters.

We will be criticized for the exponential-growth model in unit values,δ and ρ, but one must consider world population growth and - even when that subsides - there will likely be a continued growth in standard of living of individuals (per capita consumption). The exponential demand model may, in fact, be the *best* one for even a century or more into the future and that is not to say that it is a *good* model.

A more serious problem with our view of behavior modeling is that, in reality, the forest is only one asset in a portfolio of assets and the usual motivation is the short-term optimization of the entire portfolio.

An important way to improve these studies is to add state variables. State variables could be added to represent other aspects of stand dynamics such as age-dependent growth (Brewer, 1981, 1989). In this way, the important characterization of diameter change can be made more accurate. Also, this addition would remove the restriction to even-aged forest stands. There are other intriguing possibilities. For example, a variable to represent a pest population could be added and this would allow for an accounting of pest-management costs in the present-value calculation. Still another alternative is to add a state equation to describe the dynamics of a wildlife species for which the stand is habitat. In that version of the optimal silviculture problem, the proper representation of the wildlife in the present-value of the stand will introduce the usual challenges. That is to say, at the heart of every wildlife problem, there lies a difficulty with a proper accounting.

Imagine describing optimal silviculture to experienced and responsible executives: The time-scales are all wrong - a five to seven year horizon being typical for most industries. It seems that the tendency is for the time horizon of the manager to be the next stockholders meeting and the "value" of importance, in that case, is the bottom line at that time. If optimal silviculture is ever to become part of real forest management, it will be because of government leadership inspired by a commitment of society to the long-term time bioeconomic health of the forest. Elected officials have their own time scales, so the social commitment will have to be very great indeed.

Finally, we concede that, eventually, these matters will be studied with a mathematical programming technique combined with a complex model of stand growth. However, an overall perspective is obtained with that approach only after exhaustive numerical studies, if then. For the latter purpose, the type of theoretical study illustrated in this paper has no substitute.

References

Betters, D.R. and E.A Steinkamp and M.T. Turner, 1991 "Singular Path Solutions and Optimal Thinning for Even-Aged Forest Stands" Forest Science, Vol. **37**, No. 6, pp 1632-1640

J.W. Brewer, 1981 "The Bilinear, Modal State Equations For Age-Dependent Growth Control" J. of Dyn. Syst., Meas., and Contr. (ASME), Vol. **103**, No. 2, pp. 89-94

J.W. Brewer, 1989 "The Age-Dependent Eigenfunctions Of Certain Kolmogorov Equations Of Engineering-Economics And Biology" J. Appl. Math. Modelling Vol. **13**, pp. 47-57

J.W. Brewer, 1993 "Populations: Optimal Impulsive Control" Concise Encyclopedia of Environmental Systems (P.C. Young, Editor) Pergamon Press, pp 447-450

Donnelly, D.M. and D.R. Betters, 1991 "Optimal Control For Scheduling A Final Harvest In Even-Aged Forest Stands" Forest Ecology and Management, **46**: 135-149

Donnelly, D.M. and D.R. Betters and M.T. Turner and R.E, Gaines (1992) "Thinning Even-Aged Forest Stands: Behavior of Singular Path Solutions In Optimal Control Analyses" Research Paper RM-307. Fort Collins, CO: U.S. Department of Agriculture, Forest Service, Rocky Mountain Forest And Range Experiment Station. 12p

Intriligator, M., 1971 Mathematical Optimization and Economic Theory Prentice-Hall, Chapters 13, 14

Reed, W.J. and J. Apaloo, 1991 "Evaluating The Effects Of Risk On The Economics Of Juvenile Spacing And Commercial Thinning" Ca. J. For. Res., Res **21**, pp 1390-1400

Table 1: The Results Of Example 1

For Various Values of Δt

Time Between Inventories (Years)	Switch To Thinning-Type Impulses In Year	Switch To Free-Growth In Year ...	Clear-Cut And Regenerate In Year ...
1/2	40.5	59	81.5
1	40	59	81
5	40	60	80
10	40	80	100
20	40	120	140

Optimal Impulsive Control In A Model Forest

John W. Brewer

Professor Emeritus

Department of Mechanical

and Aeronautical Engineering

University of California, Davis 95616

Dennis M. Donnelly

Research Forester

Rocky Mountain Forest and Range Experiment Station

Fort Collins, Colorado, 80526